FALL 2025: MATH 830 DAILY HOMEWORK

Throughout R will denote a commutative ring.

Monday, August 18. 1. Let M be an R-module. Show that:

(i) $0 \cdot x = 0$, for $0 \in R$ and all $x \in M$.
(ii) $r \cdot 0 = 0$, for all $r \in R$ and $0 \in M$.
(iii) $-1 \cdot x = -x$, for all $x \in M$.

2. Let $\phi : A \to B$ be a R-module homomorphism. Prove that the kernel of ϕ is a submodule of A, the image of ϕ is a submodule of B, and the inverse image of C is a submodule of A, for any submodule $C \subseteq B$.

3. Suppose M is an R-module and $I \subseteq R$ is an ideal. Define IM to be the set of all finite linear combinations of the form $i_1x_1 + \cdots + i_nx_n$, with each $i_j \in I$ and $x_j \in M$.

(i) Prove that IM is a submodule of M.
(ii) Show that if $X \subseteq M$ and $\langle X \rangle = M$, then IM is the set of all finite linear combinations of the form $i_1x_1 + \cdots + i_nx_n$, with each $i_j \in I$ and $x_j \in X$.
(iii) Prove that M/IM has the structure of an R/I-module.
(iv) Conclude that if $IM = 0$, then M is also an R/I module and $N \subseteq M$ is an R-submodule of M if and only if N is an R/I submodule of M. Hence, the submodule structures of M as a module over R and as a module over R/I are the same.

Wednesday, August 20. 1. Prove the third isomorphism theorem, as stated in class.

2. Let M be an R-module and $\{H_i\}_{i\in I}$ an arbitrary collection of submodules of M. Define what it means for $M = \bigoplus_{i\in I} H_i$, the direct sum of the H_i, and then show that this is equivalent to requiring that every element $x \in M$ can be written uniquely as a sum of finitely elements of the form $h_i \in H_i$.

3. Let $H_1, \ldots, H_r \subseteq M$ be submodules. Show that $H_1 \times \cdots \times H_r$ has the natural structure of an R-module, where $H_1 \times \cdots \times H_r$ denotes the set of r-tuples of the form $(h_1, \ldots, h_r)$, with each $h_i \in H_i$. Prove that if $M = H_1 \oplus \cdots \oplus H_r$, then $M \cong H_1 \times \cdots \times H_r$.

4. Look up or review the proof that any two bases for an infinite dimension vectors space V over the field F have the same cardinality.

Friday, August 22. 1. Let $\{H_i\}_{i\in I}$ be a collection of R modules and $S := \bigoplus_{i\in I} H_i$ be the (external) direct sum of the H_i. For each $i \in I$, we have a canonical injective R-module homomorphism $j_i : H_i \to S$, given by $j_i(h) = t$, where $t \in S$ is the I-tuple whose ith component is h and all other components are 0.

(i) Suppose T is an R-module and for each $i \in I$, we have an R-module homomorphism $f_i : H_i \to T$. Show that there exists a unique R-module homomorphism $F : S \to T$ such that $Fj_i = f_i$, for all $i \in I$.
(ii) Let P be an R-module with the following property: For each $i \in I$, there exists an injective R-module homomorphisms $k_i : H_i \to P$ such that given R-module homomorphisms $g_i : H_i \to T$, there exists a unique R-module homomorphism $G : P \to T$ satisfying $Gk_i = g_i$. Prove that P is isomorphic to S.

Thus direct sums are characterized by the existence of the inclusions maps j_i satisfying the universal property given in (i).

2. Let $\{H_i\}_{i\in I}$ be a collection of R-modules and $S := \bigoplus_{i\in I} H_i$, the external direct sum. Prove that S is the internal direct sum of a collection of submodules isomorphic to the H_i.

3. Let R be a commutative ring, $n \geq 1$, and suppose $x_1, \dots, x_n \in R$ satisfy $\langle x_1, \dots, x_n \rangle = R$. Let $\phi : R^n \to R$ be defined by $\phi(\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}) = a_1x_1 + \cdots + a_nx_n$. Prove that $R^n = K \bigoplus L$, where K is the kernel of ϕ and L is a submodule of R^n isomorphic to R.

Monday, August 25. 1. Suppose $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is a short exact sequence of R-modules. Prove that if C is a free R-module, then there exists an R-module homomorphism $j : C \to B$ satisfying:

(i) $g \circ j = 1_C$.
(ii) $B = f(A) \bigoplus j(C)$.

Conclude that B is isomorphic to $A \bigoplus C$.

2. Let $A \subseteq \mathbb{Z}^3$ be the $\mathbb{Z}$-submodule of $\mathbb{Z}^3$ generated by the columns of the matrix $\begin{pmatrix} 1 & 9 & 5 \\ 2 & 6 & 4 \\ -1 & 1 & 0 \end{pmatrix}$. Find a basis for A.

3. Provide the details of the proof of the main theorem from today's lecture in the most general case, using the notation from class.

Wednesday, August 27. 1. Let M be a Noetherian R-module and $\phi : M \to M$ a surjective R-module homomorphism. Prove that ϕ is an isomorphism. Hint: Consider the kernels of the maps ϕ^i.

2. Let $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be an exact sequence or R-modules. Prove that B is Noetherian (respectively, Artinian) if and only A and C are Noetherian (respectively, Artinian). Note: To say that the sequence is exact means that f is injective, g is surjective and $\text{im}(f) = \ker(g)$.

3. Let M be a Noetherian R-module and J the annihilator of M, i.e, $J := \{r \in R \mid rx = 0 \text{ for all } x \in M\}$. Prove that R/J is a Noetherian ring (equivalently, a Noetherian R-module). Hint: Find a an R-module homomorphism from R to a finite direct sum of M with itself. Conclude that R is a Noetherian ring.

4. Assume that R has a *unique* maximal ideal P and let A be an Artinian R-module. For $p \in P$ and $x \in A$, prove there exists $n \geq 1$ such that $p^n x = 0$. Conclude that if P is finitely generated (e.g., R is Noetherian), then for each $x \in A$, there exists $r \geq 1$ (depending on x) such that $P^r x = 0$.

Friday, August 29. The following exercises lead to a proof of the fundamental fact that an Artinian ring is a Noetherian ring.

1. Let R be an Artinian ring. Prove that R has finitely many maximal ideals. (Recall that maximal ideals are prime ideals, and if $P \subseteq R$ is a prime ideal containing the product of ideals IJ, then P contains I or P contains J.)

2. Let $J \subseteq R$ be the Jacobson radical of R, i.e, J is the intersection of the maximal ideals of R. Show that $x \in J$ if and only if for all $r \in R$, $1 - rx$ is a unit in R.

3. Let R be an Artinian ring and J its Jacobson radical. Prove that $J^n = 0$, for some $n \geq 1$. Recall that J^n denotes the ideal of R generated by all n-fold products of elements of J.

4. The ring theoretic analogue of the Chinese Remainder Theorem states that if $I, J \subseteq R$ are comaximal, i.e., $I + J = R$, then $R/(I \cap J) \cong R/I \bigoplus R/J$. Use this and the previous exercises to conclude that if R is Artinian with maximal ideals $M_1, \dots, M_r$, then there exists $n \geq 1$ such that $R \cong R/M_1^n \bigoplus \cdots \bigoplus R/M_r^n$.

5. Explain why the previous exercise reduces the proof that an Artinian ring is Noetherian to the special case that R is Artinian with one maximal ideal M satisfying $M^n = 0$, for some $n \geq 1$.

6. Let R and M be as in the previous problem. Prove by induction on n that R is Noetherian. Hint: Use the exact sequences $0 \to M^{n-1}/M^n \to R/M^n \to R/M^{n-1} \to 0$ and the fact that any finite dimensional vector space over a field is both Noetherian and Artinian.

Wednesday, September 3. 1. Let M and N be simple R-modules and $\phi : M \to N$ an R-module homomorphism.

(i) Prove that either ϕ is the zero map, or ϕ is an isomorphism.
(ii) Show that the set of R-module homomorphisms from M to M is a division ring, where multiplication is given by composition.

2. Let
$$\mathcal{C}: \quad 0 \to C_n \xrightarrow{f_n} C_{n-1} \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_2} C_1 \xrightarrow{f_1} C_1 \to 0$$
be a sequence of finite length R-modules and R-module homomorphisms satisfying $f_i \circ f_{i+1} = 0$, for all i, in other words, $\mathcal{C}$ is a *complex* of finite length R-modules. For each i, define the ith *homology module* of the complex $\mathcal{C}$ to be $\mathrm{H}_i(\mathcal{C}) := \ker(f_i)/\mathrm{im}((f_{i+1})$. Prove that the homology modules $\mathrm{H}_i(\mathcal{C})$ have finite length and
$$\sum_{i\geq 0}(-1)^i\lambda(C_i) = \sum_{i\geq 0}(-1)^i\lambda(\mathrm{H}_i(\mathcal{C})).$$
The alternating sum $\sum_{i\geq 0}(-1)^i\lambda(\mathrm{H}_i(\mathcal{C}))$ is called the *Euler characteristic* of the complex $\mathcal{C}$.

Monday, September 15. These exercises show that two of the facts established in class for finitely generated modules over a PID fail if the module is not finitely generated. In particular, these show: (i) If M is not finitely generated over the PID R, then $T(M)$ need not be a direct summand of M and (ii) An arbitrary torsion-free module over a PID need not be free. We take the case $R := \mathbb{Z}$. Let $\mathcal{P}$ denote the set of prime numbers in $\mathbb{Z}$ and set $M := \prod \mathbb{Z}_{p\in\mathcal{P}}$, the *direct product* of all $\mathbb{Z}_p$.

1. Show that $T(M) = \bigoplus_{p\in\mathcal{P}} \mathbb{Z}_p$.

2. Show that $\bigcap_{p\in\mathcal{P}} pM = 0$, and thus $\bigcap_{p\in\mathcal{P}} pN = 0$, for any submodule $N \subseteq M$.

3. Set $x := (1, 1, 1, 1, \cdots) \in M$. Show that the image of x in $M/T(M)$ is not zero.

4. Show that the image of x in $M/T(M)$ belongs to $\bigcap_{p\in\mathcal{P}} p(M/T(M))$.

5. Conclude: (i) $T(M)$ is not a direct summand of M and (ii) $M/T(M)$ is torsion-free, but not free.

Wednesday, September 17. 1. Suppose R is a PID and $M = \langle x\rangle \oplus \langle y\rangle$ with non-zero x, y satisfying $\mathrm{ann}(x) = aR$, $\mathrm{ann}(y) = bR$, and GCD $(a, b) = 1$. Show that $\mathrm{ann}(x + y) = abR$.

2. Under the assumptions in Problem 1, show that $M = \langle x + y\rangle$. Hint: Adapt the proof of the Chinese remainder theorem.

3. Let A be an $n \times m$ matrix over the commutative ring R, write K for the submodule of R^n generated by the columns of A and set $M := R^n/K$. For an invertible $n \times n$ matrix P and invertible $m \times m$ matrix Q, set $\tilde{A} := P^{-1}AQ$. Let $\tilde{K}$ be the submodule of R^n generated by the columns of $\tilde{A}$ and set $\tilde{M} := R^n/\tilde{K}$. Show that M is isomorphic to $\tilde{M}$. Hint: Consider the maps $R^m \xrightarrow{A} R^n$ and $R^m \xrightarrow{\tilde{A}} R^n$, and think in terms of change of bases for R^n and R^m.

Friday, September 19. 1. Let M be an R-module and consider the exacts sequences
$$(*) \quad 0 \to P_n \xrightarrow{f_n} P_{n-1} \xrightarrow{f_{n-1}} \cdots \to P_1 \xrightarrow{f_1} P_0 \xrightarrow{\pi} M \to 0$$
and
$$(**) \quad \cdots \to P'_{n+1} \xrightarrow{g_{n+1}} P'_n \xrightarrow{g_n} P'_{n-1} \xrightarrow{g_{n-1}} \cdots \to P'_1 \xrightarrow{g_1} P'_0 \xrightarrow{\pi} M \to 0$$
Prove that there exist projective R-modules $Q_0, Q'_0, \ldots, Q_n, Q'_n$ and homomorphisms $\tilde{f}_j, \tilde{g}_j$ such that the sequences
$$(*) \quad 0 \to P_n \oplus Q_n \xrightarrow{\tilde{f}_n} P_{n-1} \oplus Q_{n-1} \xrightarrow{\tilde{f}_{n-1}} \cdots \to P_1 \oplus Q_1 \xrightarrow{\tilde{f}_1} P_0 \oplus Q_0 \xrightarrow{\pi} M \to 0$$
$$(**) \quad \cdots \to P'_{n+1} \xrightarrow{g_{n+1}} P'_n Q'_n \xrightarrow{\tilde{g}_n} P'_{n-1} \oplus Q'_{n-1} \xrightarrow{\tilde{g}_{n-1}} \cdots \to P'_1 \oplus Q'_1 \xrightarrow{\tilde{g}_1} P'_0 \oplus Q'_0 \xrightarrow{\pi} M \to 0$$
are exact and $P_j \oplus Q_j \cong P'_j \oplus Q'_j$, for all $0 \leq j \leq n$.

2. Prove the following variation of Nakayama's lemma: Let M be a finitely generated R-module and $J \subseteq R$ a proper ideal. If $JM = M$, then there exists $j \in J$ such that $(1 + j) \cdot M = 0$.

Monday, September 22. 1. Let $S \subseteq R$ be a multiplicatively closed subset and M an R-module. For $(m, s), (m', s')$ in $M \times S$, defined $(m, s) \sim (m', s)'$ if there exists $s'' \in S$ such that $s''(s'm - sm') = 0$.

(i) Show the relation defined above is an equivalence relation.
(ii) Writing m/s for the equivalence class of (m, s) let M_S denote the set of all such equivalence classes and prove that M_S has a well-defined structure as an R-module.

2. Let $S \subseteq R$ be a multiplicatively closed set. Let $\phi : R \to R_S$ be the canonical ring homomorphism taking $r \in R$ to $r/1 \in R_S$.
(i) Describe the kernel of ϕ.
(ii) For an ideal $I \subseteq R$, show that I_S is an ideal of R_S. Show that P_S is a prime ideal, if $P \subseteq R$ is a prime ideal.
(iii) Let $J \subseteq R_S$ be an ideal. Describe the ideal $\phi^{-1}(J)$.
(iv) Show that for an ideal $J \subseteq R_S$, $\phi^{-1}(J)_S = J$. Thus, every ideal $J \subseteq R_S$ is of the form I_S, for some ideal $I \subseteq R$.
(v) Give an example to show that for an ideal $I \subseteq R$, $\phi^{-1}(I_S)$ can strictly contain I.
(vi) Show that if $P \subseteq R$ is a prime ideal, and $P \cap S = \emptyset$, then $\phi^{-1}(P_S) = P$.
(vii) Conclude that there is a 1-1 correspondence between the prime ideals of R disjoint from S and the prime ideals of R_S.

Wednesday, September 24. Let $S, T \subseteq R$ be multiplicatively closed sets.

1. Give an example of an ideal I contained in a ring R with multiplicatively closed set S such that $\phi^{-1}(I_S)$ properly contains I, where $\phi : R \to R_S$ is the canonical map. In particular, it is possible to have $I_S = J_S$ for ideals $I, J \subseteq R$, yet $I \neq J$.

2. Show that ST is a multiplicatively closed subset of R and that the rings R_{ST} and $(R_S)_{T'}$ are isomorphic, where $T' := \{\frac{t}{1} \in R_S \mid t \in T\}$.

3. Given an exact sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ of R-modules, prove that the the induced sequence of R_S-modules $0 \to A_S \xrightarrow{f_S} B_S \xrightarrow{g_S} C_S \to 0$ is exact.

Friday, September 26. Throughout, all modules A, B, C, M are R-modules and all maps are R-module homomorphisms.

1. Given $A \xrightarrow{f} B$, show there are induced maps:
(i) $\mathrm{Hom}_R(B, M) \xrightarrow{f^*} \mathrm{Hom}_R(A, M)$
(ii) $\mathrm{Hom}_R(M, A) \xrightarrow{\hat{f}} \mathrm{Hom}_R(M, B)$.

2. Given a short exact sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$, show that there are exact sequences
(i) $0 \to \mathrm{Hom}_R(C, M) \xrightarrow{g^*} \mathrm{Hom}_R(B, M) \xrightarrow{f^*} \mathrm{Hom}_R(A, M)$
(ii) $0 \to \mathrm{Hom}_R(M, A) \xrightarrow{\hat{f}} \mathrm{Hom}_R(M, B) \xrightarrow{\hat{g}} \mathrm{Hom}_R(M, C)$.

3. Assume the short exact sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ splits. Prove that f^* in 2(i) and $\hat{g}$ in 2(ii) are surjective. In other words, the given exact sequence remains exact upon applying $\mathrm{Hom}_R(-, M)$ and $\mathrm{Hom}_R(M, -)$.

Monday, September 29. Use Baer's Criterion to work the following problems.

1. Show that $\mathbb{Z}_{p^\infty}$ is an injective $\mathbb{Z}$-module, where $\mathbb{Z}_{p^\infty}$ is the set of elements in $\mathbb{Q}/\mathbb{Z}$ annihilated by some power of p.

2. For $n \geq 2$, show that $\mathbb{Z}_n$ is not an injective $\mathbb{Z}$-module, but it is an injective module over the ring $\mathbb{Z}_n$.

3. Let R be an integral domain with quotient field K. Show that K is an injective R-module.

Wednesday, October 1. 1. Let $\{Q_i\}_{i \in I}$ be a family of R-modules. Show that $\prod_{i \in I} Q_i$ is an injective R-module if and only if each Q_i is an injective R-module.

2. A theorem of H. Bass states that the ring R is Noetherian if and only every direct sum of injective modules is injective. Use Baer's Criterion to prove part of Bass's Theorem, namely: Let R be a Noetherian ring, and

$\{Q_\alpha\}_{\alpha\in A}$ a collection of injective R-modules. Then $\bigoplus_{\alpha\in A} Q_\alpha$ is injective. Hint: You must show that given an ideal $I \subseteq R$, any diagram

$$\begin{tikzcd} 0 \arrow[r] & I \arrow[r, "i"] \arrow[d, "g"] & R \arrow[dl, dotted, "\rho"'] \\ & \bigoplus_{\alpha\in A} Q_\alpha & \end{tikzcd}$$

can be completed. Now use the fact that I is finitely generated.

3. Let $\{P_i\}_{i\in I}$ be a family of R-modules. Show that $\bigoplus_{i\in I} P_i$ is a projective R-module if and only if each P_i is a projective R-module.

Wednesday, October 15. 1. Show that the direct limit of projective modules need not be projective by showing that as a $\mathbb{Z}$-module, $\mathbb{Q}$ is a direct limit of free $\mathbb{Z}$-modules, but is not a projective $\mathbb{Z}$-module.

2. Give a rigorous proof that $k[[x]]$ is the inverse limit of the ring $k[x]/\langle x^n\rangle$.

3. Show that over a Noetherian ring, the direct limit of injective modules is injective. [1]

Friday, October 17. 1. Suppose $\mathrm{id}_R(M) = d$. Use an injective version Schanuel's Lemma to prove that in any injective resolution of M, the $(d-1)^{\text{st}}$ cokernel is injective.

2. Let $\{A_i\}_{i\in I}$, $\{B_i\}_{i\in I}$, A and B be

(i) Show that $\mathrm{Hom}_R(\bigoplus_{i\in I} A_i, B) \cong \prod_{i\in I} \mathrm{Hom}_R(A_i, B)$.
(ii) Use (i) to show that $\mathrm{Ext}^n_R(\bigoplus_{i\in I} A_i, B) \cong \prod_{i\in I} \mathrm{Ext}^n_R(A_i, B)$, for all $n \geq 1$.
(iii) Formulate and prove versions of (i) and (ii) for $\mathrm{Hom}_R(A, \bigoplus_{i\in I} B_i)$ and $\mathrm{Ext}^n_R(A, \bigoplus_{i\in I} B_i)$.

3. Let A, B be R-modules with $I := \mathrm{ann}(A)$ and $J := \mathrm{ann}(B)$. Prove that $I + J \subseteq \mathrm{ann}(\mathrm{Ext}^n_R(A, B))$, for all $n \geq 0$.

4. Let $\phi : R^n \to R^m$ be an R-module homomorphism. First show that there is an $m \times n$ matrix A such that $\phi(v) = Av$, for all $v \in R^n$. Here were are writing the elements of R^n and R^m as column vectors. Then show that the induced map $\phi^* : \mathrm{Hom}_R(R^m, R) \to \mathrm{Hom}_R(R^n, R)$ is multiplication by A^t, the transpose of A.

5. For $R := k[x, y]$,the polynomial ring in two variables over the field k, calculate $\mathrm{Ext}^n_R(k, R)$, for all $n \geq 0$.

Monday, October 20. 1. Find an injective resolution of $\mathbb{Z}_n$ as a $\mathbb{Z}$-module and then use it to calculate $\mathrm{Ext}^r_{\mathbb{Z}}(\mathbb{Z}_m, \mathbb{Z}_n)$, for all $r \geq 0$.

2. For the ring $R = \mathbb{Z}_n$, show that R is an injective R-module.

Wednesday, October 22. State and prove an injective version of the proposition given in today's lecture.

Monday, Wednesday, October 27. 1. Use the results and techniques from today's lecture to show that $\mathrm{Ext}^3_R(A, B)$ is independent of the projective resolution of A, the injective resolution of B and that the corresponding cohomology modules are isomorphic.

2. Prove that a $\mathbb{Z}$-module M is torsion-free if and only if $\mathrm{Ext}^1_{\mathbb{Z}}(M, \mathbb{Z})$ is divisible.

Wednesday, October 29. 1. Calculate $\mathbb{Z}_n \otimes_{\mathbb{Z}} \mathbb{Z}_m$, for $n, m \geq 1$.

2. Prove that $R/I \otimes_R M \cong M/IM$, for $I \subseteq R$ an ideal and M an R-module. Conclude that for ideals $I, J \subseteq R$, $(R/I) \otimes_R (R/J) \cong R/(I + J)$.

3. For R-modules M, N, L prove that $(M \otimes_R N) \otimes_R L \cong M \otimes_R (N \otimes_R L)$.

Friday, October 31. 1. Let R be an integral domain with ideals $I, J \subseteq R$. Let $0 \to I \xrightarrow{i} R$ be the natural inclusion. Prove:

(i) The image of the induced map $I \otimes_R J \xrightarrow{i\otimes 1_J} R \otimes J = R$ is the ideal IJ.
(ii) The kernel of the map $I \otimes_R J \xrightarrow{i\otimes 1_J} IJ$ is the torsion submodule of $I \otimes_R J$. For this, you should use property (x) of tensor product.

[1]Interesting Fact: Every module is the inverse limit of injective modules!

2. Prove that if we tensor a short exact sequence of R-modules with a projective R-module, then the resulting sequence is also a short exact sequence.

Monday, November 3. For these exercises you may assume that $\mathrm{Tor}_n^R(A,B)$ modules are well defined and can be calculated by first resolving A and tensoring with B or resolving B and tensroing with A.

1. Let A, B be R-modules and $S \subseteq R$ a multiplicatively closed set. Prove that there is an isomorphism of R_S-modules $\mathrm{Tor}_n^R(A,B)_S \cong \mathrm{Tor}_n^{R_S}(A_S, B_S)$, for all $n \geq 1$. You may use the fact that $(A \otimes_R B)_S \cong A_S \otimes_{R_S} B_S$, as R_S-modules.

2. Calculate $\mathrm{Tor}_n^{\mathbb{Z}_{48}}(\mathbb{Z}_{12}, \mathbb{Z}_{16})$ in two ways, for all $n \geq 0$.

Wednesday, November 5. 1. For the example in today's lecture where $R = k[x,y]/\langle xy \rangle$, calculate $\mathrm{Tor}_n^R(A,B)$ for $n = 3, 4, 5$ by using the given resolution of $R/\mathfrak{m}$.

2. For a collection of R-modules $\{A_i\}_{i \in I}$, B prove that $\mathrm{Tor}_n(\bigoplus_{i \in I} A_i, B) \cong \bigoplus_{i \in I} \mathrm{Tor}_n^R(A_i, B)$, for all $n \geq 1$.

Monday, November 17. Let R be a Noetherian ring, $I \subseteq R$ an ideal amd M an R-module. The jth local cohomology module of R with respect to I, denoted by $\mathrm{H}_I^j(M)$ is the jth cohomology module obtained by applying the functor $\Gamma_I(-)$ to a deleted injective resolution of M, where for any R-module A,
$\Gamma_I(A) = \{a \in A \mid I^n a = 0 \text{ for some } n \geq 1\}$.

1. Show that:
   (i) $\mathrm{H}_I^0(M) = \Gamma_I(M)$.
   (ii) If M is torsion free, then $\mathrm{H}_I^0(M) = 0$ and if M has injective dimension d, $\mathrm{H}_I^j(M) = 0$, for $j > d$.
   (iii) If $0 \to M \to Q \to C \to 0$ is an exact sequence with Q injective, then $\mathrm{H}_I^{j+1}(M) = \mathrm{H}_I^i(C)$, for all $j \geq 1$.

2. Suppose Q is an injective R-module. Show that $\Gamma_I(Q)$ is an injective R-module. Note: This is not necessarily true when R is not Noetherian.

Wednesday, November 19. Suppose R is a Noetherian ring.

1. Let $\{M_i\}_{i \in I}$ be a collection of R modules and set $M := \bigoplus_{i \in I} M_i$. Show that $\mathrm{Ass}_R(M) = \bigcup_{i \in I} \mathrm{Ass}_R(M_i)$.

2. Let M be an R-module, $I \subseteq R$ an ideal, and assume every element of M is annihilated by a power of I. Let $N \subseteq M$ be the elements of M annihilated by I. Prove that $\mathrm{Ass}_R(M) = \mathrm{Ass}_R(N)$.

3. Let M be a finitely generated R-module and $P \subseteq R$ a prime ideal minimal over the annihilator of M. Prove that M_P has finite length as an R_P-module and that its length equals the number of times R/P appears in any prime filtration of M.

Friday, November 21. 1. For R-modules $N \subseteq M$, show that the following are equivalent:
   (i) $L \cap N \neq 0$, for all submodules $L \subseteq M$.
   (ii) Every non-zero element in M has a non-zero multiple in N.
   (iii) For an R-module homomorphism $\phi : M \to A$, if $\phi|_N$ is injective, then ϕ is injective.

If these conditions hold, we say that M is an *essential* extension of N.

2. Prove the following statements:
   (i) For modules $L \subseteq N \subseteq M$, M is an essential extension of L if and only if N is an essential extension of L and M is an essential extension of N.
   (ii) Suppose $N \subseteq L_i \subseteq M$ with $\{L_i\}_{i \in I}$ a collection of submodules of M containing N satisfying $\bigcup_{i \in I} L_i = M$. Then M is an essential extension of N if and only each L_i is an essential extension of N.
   (iii) Given $N \subseteq M$, there exists a submodule $N \subseteq L \subseteq M$, such that L is maximal with respect to being an essential extension of N in M.

Recall that every R-module A is contained in an injective R-module. For an R-module M, with $M \subseteq Q$, with Q injective, a maximal essential extension of M in Q is called an *injective envelope* of M, denoted $E(M)$.

3. Prove an injective envelop of M is an injective R-module and any two injective envelopes of M are isomorphic. Hint: First note that an injective module has no essential extensions.

Monday, November 24. 1. Suppose R is a $\mathbb{Z}$-graded ring that is not a field. Show that if R and (0) are the only homogeneous ideals, then $R \cong k[t, t^{-1}]$, the Laurent polynomial ring in one variable over a field k. Hint: The element in R corresponding to t will be homogeneous of some degree greater than zero, and will be transcendental over R_0, which you must show is a field.

2. Suppose R is a $\mathbb{Z}$-graded ring and $P \subseteq R$ is a prime ideal. Prove that there are no prime ideals properly between P^* and P. Hint: Use the previous problem.